Existence of optimal strategies in models with several consumption goods

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(Abstract)

The paper deals with the existence of optimal strategies when preferences are characterized by a constant rate of time discount ρ, by a constant elasticity of substitution σ, and homotheticity among the consumption goods available at the same time. As a consequence the utility function must be

$$U = \int_0^\infty e^{-\rho t} \frac{\nu\left(c_{1t}, c_{2t}, \ \dots \ , c_{kt}\right)^{1-\sigma} - 1}{1-\sigma} dt$$

where $\nu\left(c_{1t}, c_{2t}, \ \dots \ , c_{kt}\right)$ is an homogeneous concave function, c_{it} is the consumption of commodity *i* at time *t*, the first *k* commodities are consumption goods, the other (eventual) $n-k$ are pure capital goods.

The paper consists of 6 sections. Section 2 analyzes an economy where technology, whose returns for scale are constant, is such that the supremum growth rates of the consumption goods are identified and are $\Gamma_1, \Gamma_2, \Gamma_3, \dots, \Gamma_k$, respectively. No other information about technology will be considered. Section 3 analyzes an economy where technology is linear, but decomposable, so that the supremum growth rates of the consumption goods are different. Section 4 analizes an economy where there exist n commodities: $1, 2, \dots, n$. Commodity n is a pure capital good. The stock of commodity n is $K(t)$ and

$$\dot{K}(t) = AK_n(t) - \delta K(t)$$

where $K_n(t)$ is the amount of $K(t)$ employed in sector n, A is a positive constant, and δ is the depriciation rate, assumed to be constant over time (depriciation by evaporation). Commodities $1, 2, \dots, n-1$ are pure consumption goods. Production of consumption good i requires a scarce resource, labour, and is described through a Cobb-Douglas production function

$$C_i(t) = M_i(t) L_i(t)^{\alpha_i} K_i(t)^{1-\alpha_i}$$

where $M_i(t)$ is a given function and describes the exogeneous technical progress, $L_i(t)$ is the ammount of labour employed in sector i, and $K_i(t)$ is the ammount of capital, i.e. commodity n, employed in sector i. Obviously $\sum_{i=1}^{n-1} L_i(t) = L(t)$ and $\sum_{i=1}^{n} K_i(t) = K(t)$. It is assumed that $M_i(t) = M_{i0} e^{m_i t}$ and $L(t) = L_0 e^{\lambda t}$ with $m_1, m_2, \dots, m_{n-1}, \lambda$ given non-negative constants. It is also

assumed that $0 < \alpha_i \leq 1$ and $\prod_{i=1}^{n-1} \alpha_i < 1$. It is easily recognized that the supremum growth rates of the consumption goods are $\Gamma_i = m_i + (1-\alpha_i)(A-\delta-\lambda)$.

In all these economies optimal strategies exist if

$$\Gamma_\nu > \frac{\Gamma_\nu - \rho}{\sigma}$$

and do not exist if

$$\Gamma_\nu < \frac{\Gamma_\nu - \rho}{\sigma}$$

provided that $0 < \lim_{t\to\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t},\ e^{\Gamma_2 t},\ \dots,\ e^{\Gamma_k t}\right) < \infty$, where Γ_ν is an appropriate average of the supremum growth rates of the consumption goods, defined by function $\nu(c_{1t}, c_{2t},\ \dots, c_{kt})$:

$$\Gamma_\nu = \inf\left\{\eta \in {}^\circ\ : \lim_{t\to\infty} e^{-\eta t} \nu\left(e^{\Gamma_1 t},\ e^{\Gamma_2 t},\ \dots,\ e^{\Gamma_k t}\right) = 0\right\} = \sup\left\{\eta \in {}^\circ\ : \lim_{t\to\infty} e^{-\eta t} \nu\left(e^{\Gamma_1 t},\ e^{\Gamma_2 t},\ \dots,\ e^{\Gamma_k t}\right) = \infty\right\}$$

More dubious are the cases in which either $\lim_{t\to\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t},\ e^{\Gamma_2 t},\ \dots,\ e^{\Gamma_k t}\right) = 0$ or $\lim_{t\to\infty} e^{-\Gamma_\nu t} \nu\left(e^{\Gamma_1 t},\ e^{\Gamma_2 t},\ \dots,\ e^{\Gamma_k t}\right) = \infty$ or $\Gamma_\nu (1-\sigma) - \rho = 0$.

Section 5 analizes the contributions by Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008). Both Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008) study multisector models with exogenous progress in which the rates of technical change differ across sectors (although Acemoglu and Guerrieri use a device: they consider a single consumption good which is produced instantaneously by a number of commodities by means of an homogeneous of degree 1 production function) and both need in principle an existence condition. We will show that the trasversality condition mentioned by Acemoglu and Guerrieri (2008) consists exactly in our general condition in their very specific assumptions whereas in the specific assumptions of the model of Ngai and Pissarides (2007) the existence condition is trivially satisfied since it collapse into $\rho > 0$.